The SYK models of non-Fermi liquids and black holes

QMATH13: Mathematical Results in Quantum Physics,
Georgia Tech, Atlanta, October 9, 2016

Subir Sachdev

Talk online: sachdev.physics.harvard.edu

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

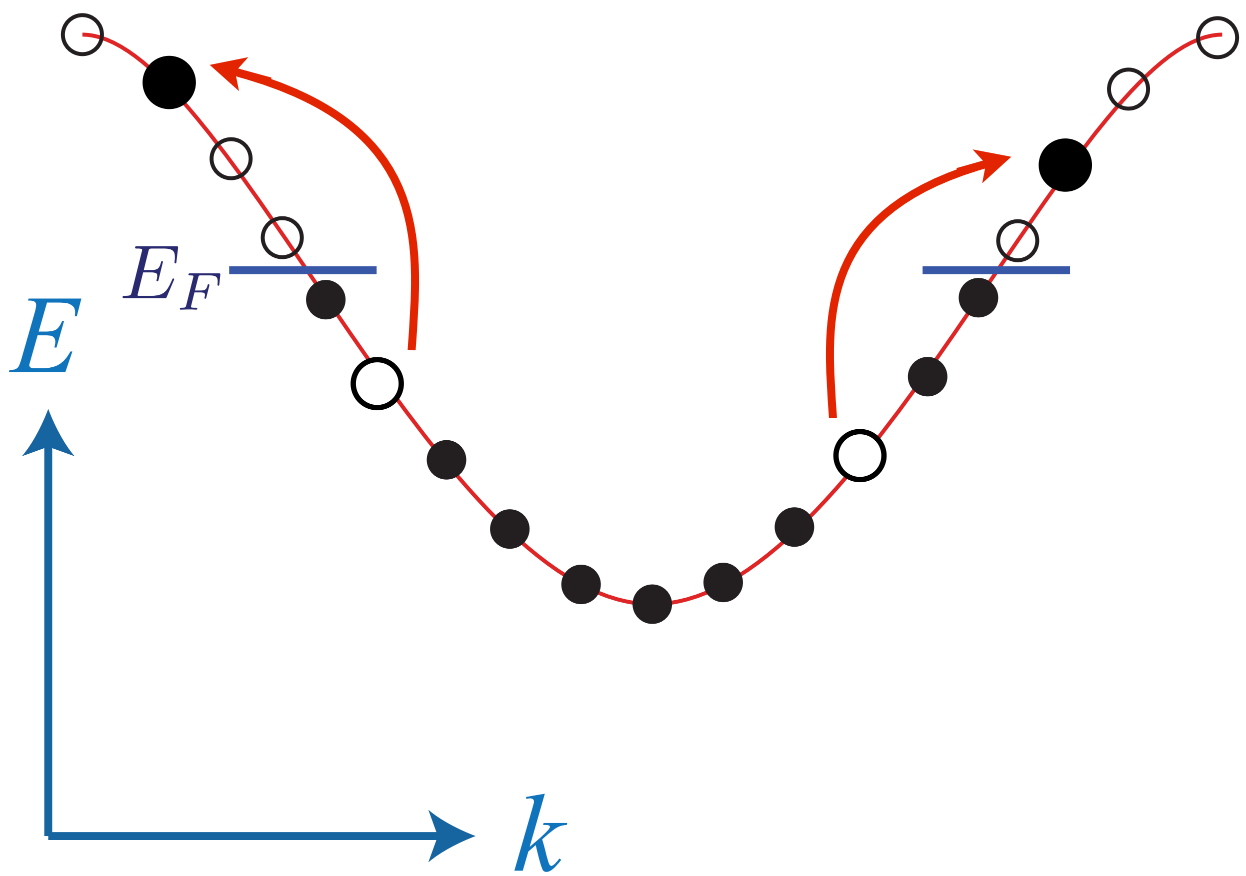
PHYSICS
VE RI TAS
HARVARD

Conventional quantum matter:

1. Ground states connected adiabatically to independent electron states

2. Boltzmann-Landau theory of quasiparticles

Metals



Luttinger's theorem: volume enclosed by the Fermi surface = density of all electrons (mod 2 per unit cell). Obeyed in overdoped cuprates

Topological quantum matter:

1. Ground states disconnected from independent electron states: many-particle entanglement

2. Boltzmann-Landau theory of quasiparticles

(a) The fractional quantum Hall effect: the ground state is described by Laughlin's wavefunction, and the excitations are *quasiparticles* which carry fractional charge.

(b) The pseudogap metal: proposed to have electron-like quasiparticles but on a "small" Fermi surface which does not obey the Luttinger theorem.

Quantum matter without quasiparticles:

1. Ground states disconnected from independent electron states: many-particle entanglement

2. No quasiparticles

Strange metals:

Such metals are found, most prominently, near optimal doping in the the cuprate high temperature superconductors.

But how can we be sure that no quasiparticles exist in a given system? Perhaps there are some exotic quasiparticles inaccessible to current experiments……..

Local thermal equilibration or phase coherence time, τ_φ:

- There is an *lower bound* on τ_φ in all many-body quantum systems of order $\hbar/(k_B T)$,

$$\tau_\varphi > C \frac{\hbar}{k_B T},$$

  and the lower bound is realized by systems *without* quasiparticles.

- In systems *with* quasiparticles, τ_φ is parametrically larger at low T;
  e.g. in Fermi liquids $\tau_\varphi \sim 1/T^2$,
  and in gapped insulators $\tau_\varphi \sim e^{\Delta/(k_B T)}$ where Δ is the energy gap.

S. Sachdev, *Quantum Phase Transitions*, Cambridge (1999)

A bound on quantum chaos:

- The time over which a many-body quantum system becomes "chaotic" is given by $\tau_L = 1/\lambda_L$, where λ_L is the "Lyapunov exponent" determining memory of initial conditions. This LYAPUNOV TIME obeys the rigorous lower bound

$$\tau_L \geq \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

A. I. Larkin and Y. N. Ovchinnikov, JETP **28**, 6 (1969)

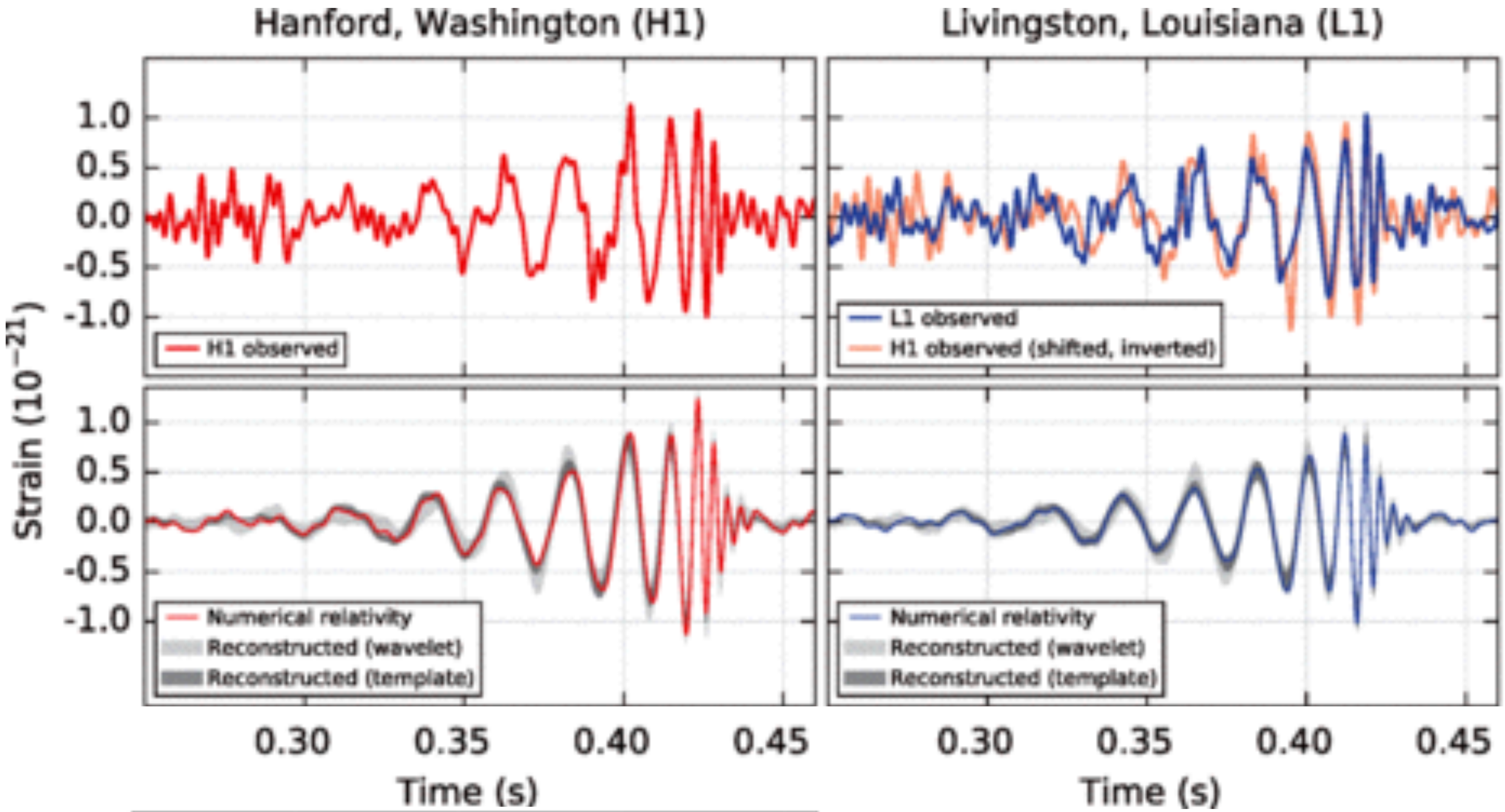
J. Maldacena, S. H. Shenker and D. Stanford, arXiv:1503.01409

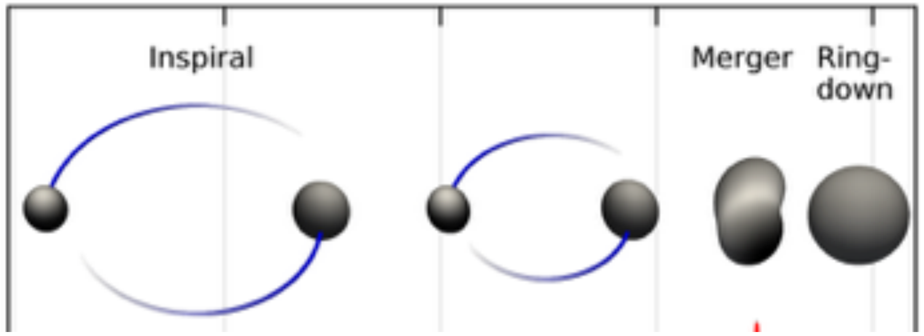
A bound on quantum chaos:

- The time over which a many-body quantum system becomes "chaotic" is given by $\tau_L = 1/\lambda_L$, where λ_L is the "Lyapunov exponent" determining memory of initial conditions. This LYAPUNOV TIME obeys the rigorous lower bound

$$\tau_L \geq \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

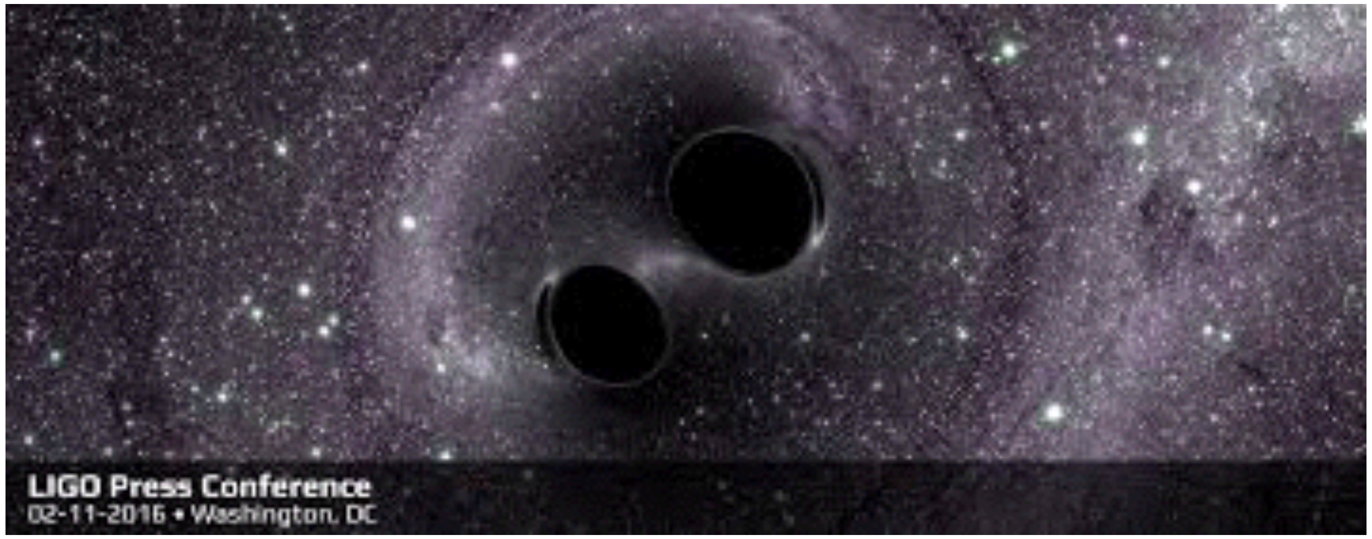
Quantum matter without quasiparticles
$\approx$ fastest possible many-body quantum chaos

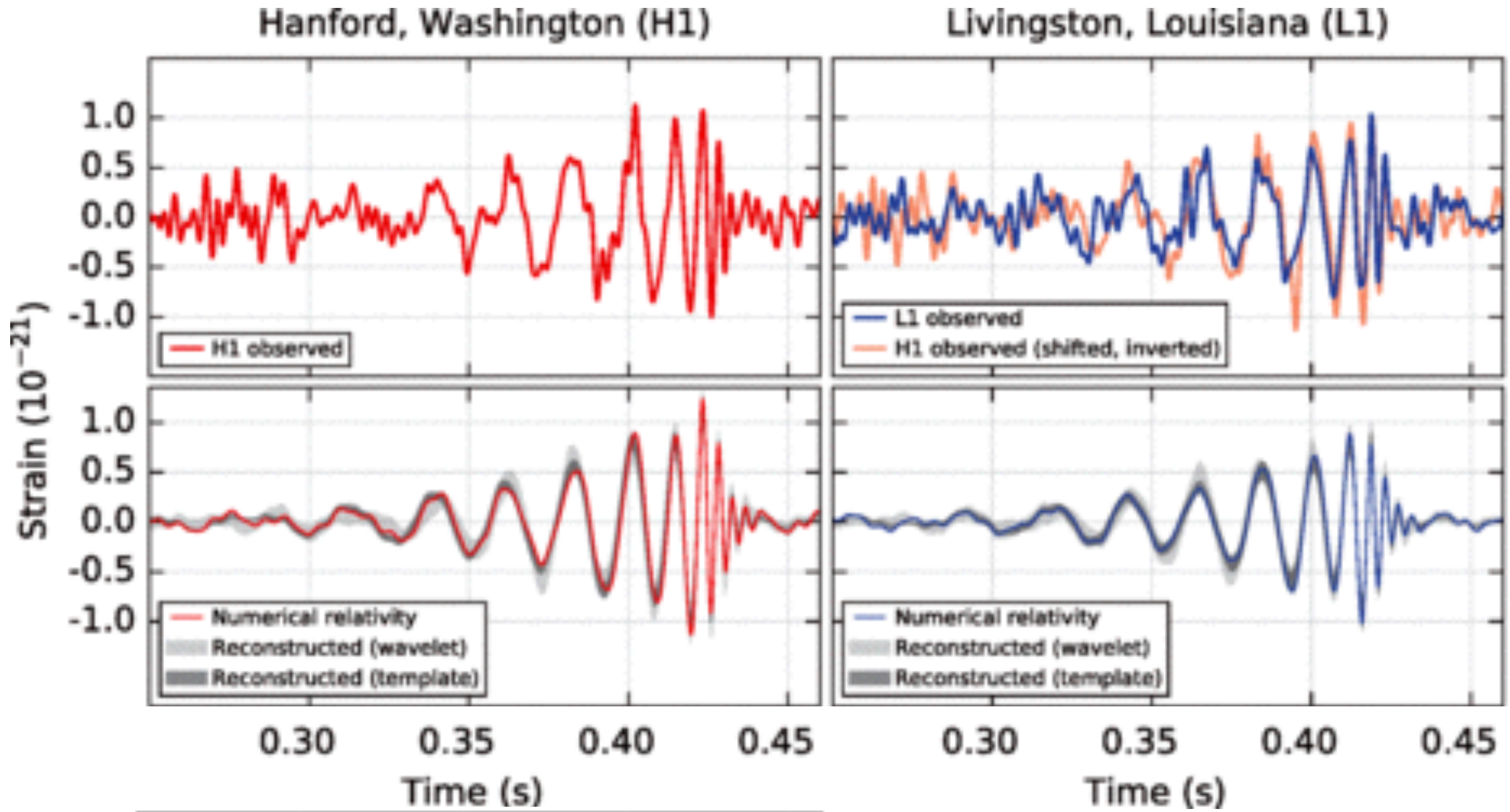


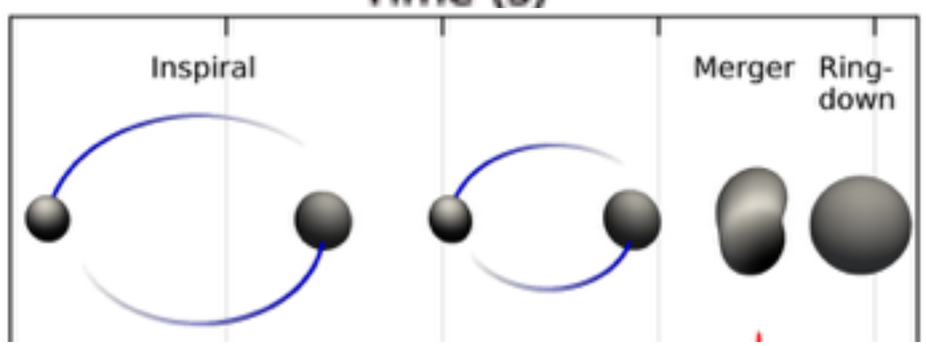


LIGO

September 14, 2015



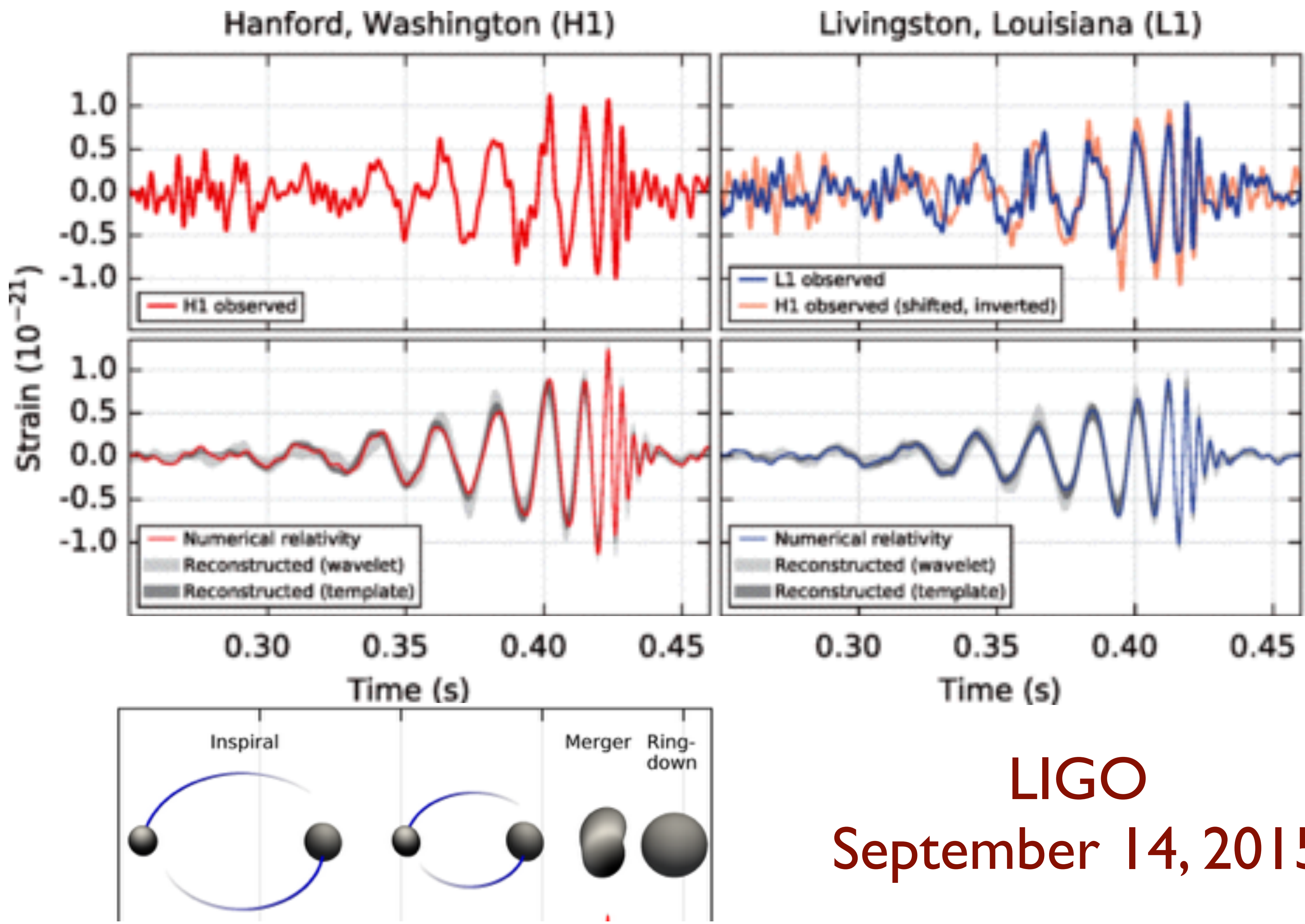




LIGO
September 14, 2015

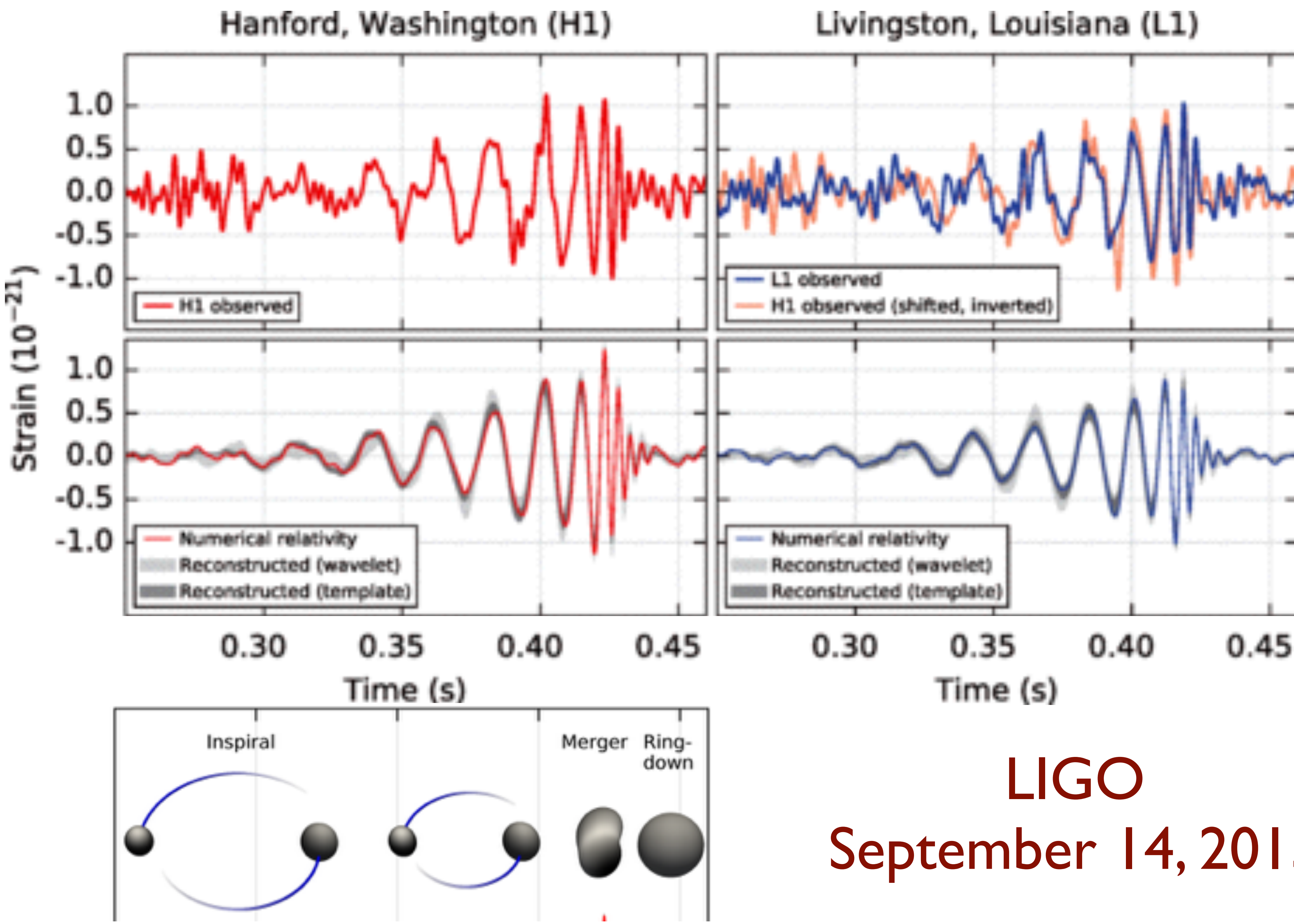
- Black holes have a "ring-down" time, τ_r, in which they radiate energy, and stabilize to a 'featureless' spherical object. This time can be computed in Einstein's general relativity theory.

- For this black hole $\tau_r = 7.7$ milliseconds. (Radius of black hole $=$ 183 km; Mass of black hole $=$ 62 solar masses.)



LIGO
September 14, 2015

- 'Featureless' black holes have a Bekenstein-Hawking entropy, and a Hawking temperature, T_H.



LIGO
September 14, 2015

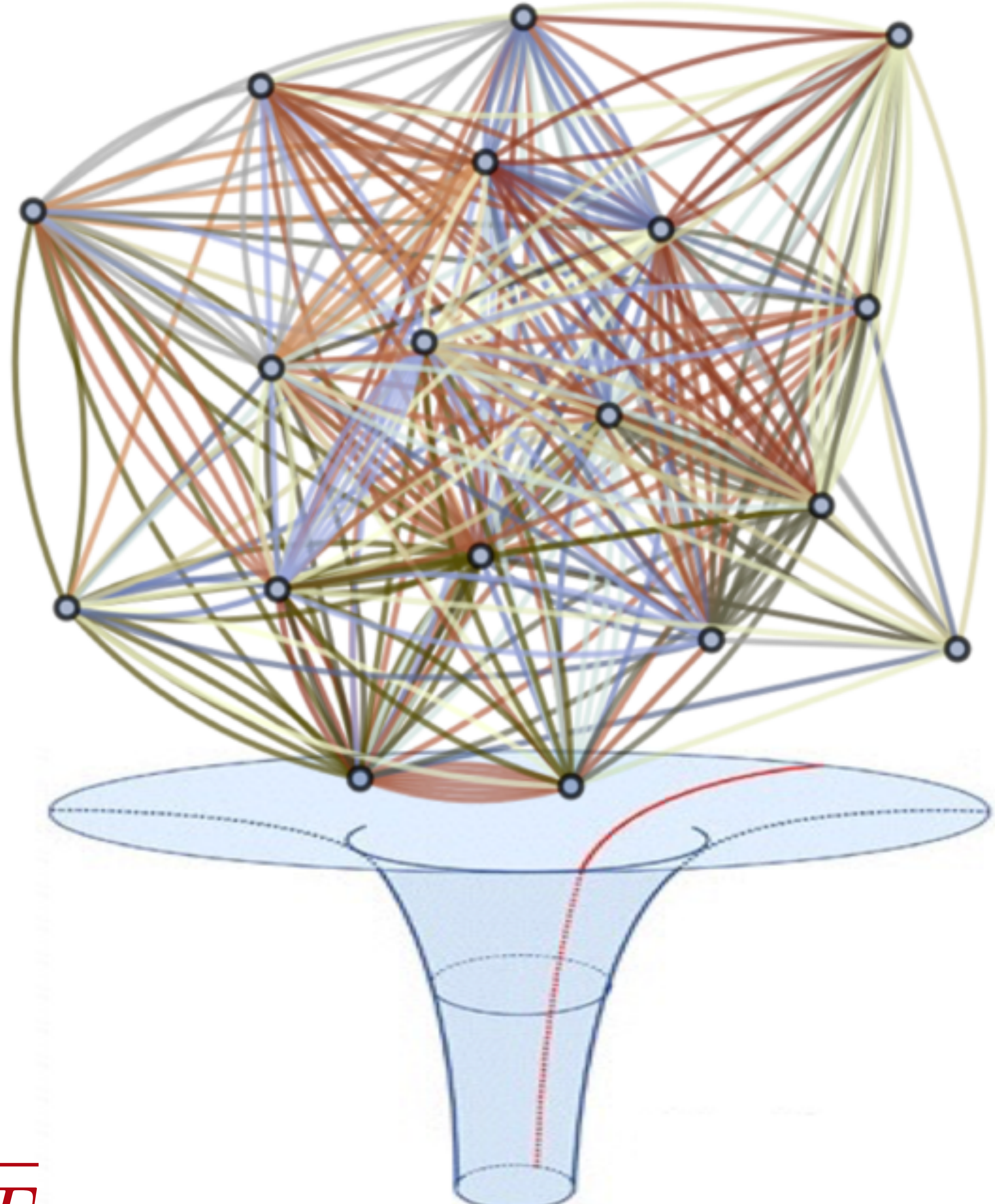
- Expressed in terms of the Hawking temperature, the ring-down time is $\tau_r \sim \hbar/(k_B T_H)$!

- For this black hole $T_H \approx 1$ nK.

The Sachdev-Ye-Kitaev (SYK) model:

- A theory of a strange metal
- Has a dual representation as a black hole
- Fastest possible quantum chaos with $\tau_L = \dfrac{\hbar}{2\pi k_B T}$

Figure credit: L. Balents



Infinite-range model with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \ldots$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

Fermions occupying the eigenstates of a *N* x *N* random matrix

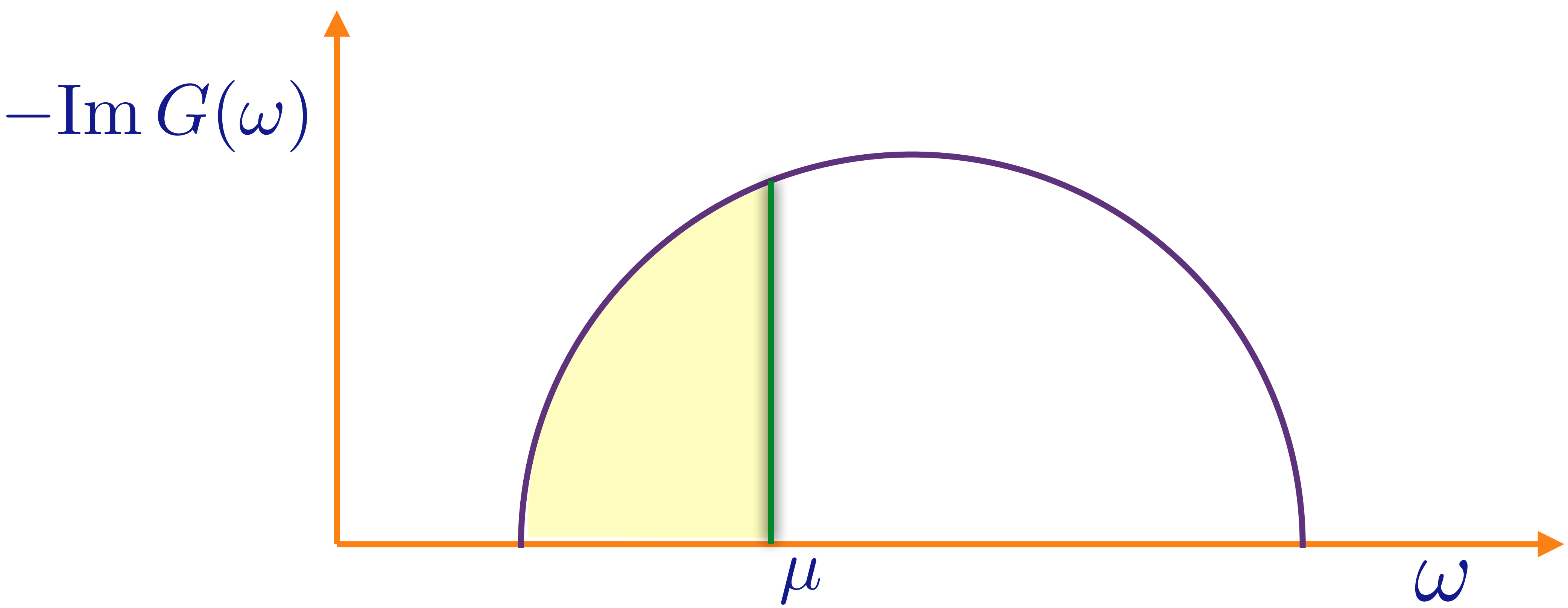
Infinite-range model with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

$G(\omega)$ can be determined by solving a quadratic equation.



Infinite-range model with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$J_{ij;k\ell}$ are independent random variables with $\overline{J_{ij;k\ell}} = 0$ and $\overline{|J_{ij;k\ell}|^2} = J^2$. We compute the lifetime of a quasiparticle, τ_α, in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamitonian with energy E_α. By Fermi's Golden rule, for E_α at the Fermi energy

$$\frac{1}{\tau_\alpha} = \pi J^2 \rho_0^3 \int dE_\beta dE_\gamma dE_\delta f(E_\beta)(1 - f(E_\gamma))(1 - f(E_\delta))\delta(E_\alpha + E_\beta - E_\gamma - E_\delta)$$
$$= \frac{\pi^3 J^2 \rho_0^3}{4} T^2$$

where ρ_0 is the density of states at the Fermi energy.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

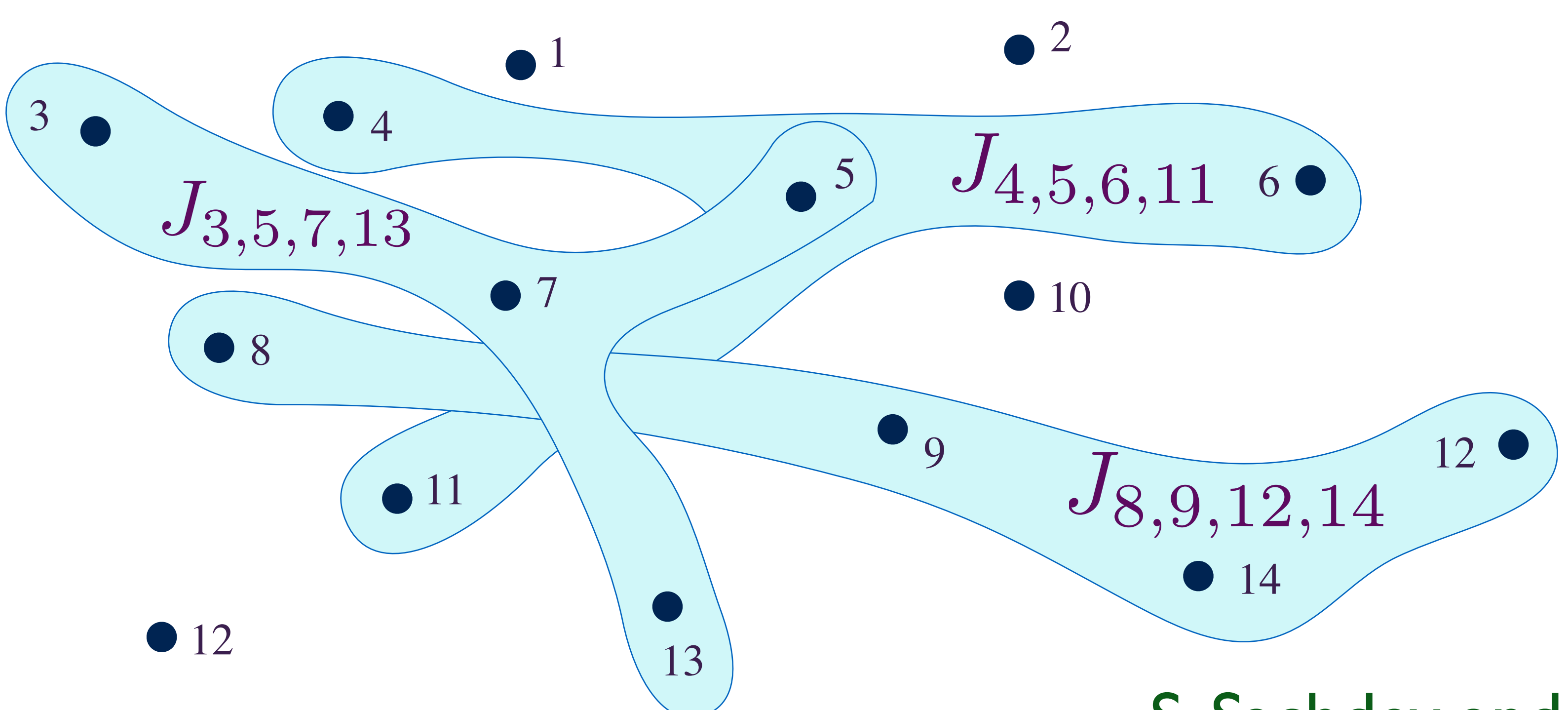
SYK model

To obtain a non-Fermi liquid, we set $t_{ij} = 0$:

$$H_{\rm SYK} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$\mathcal{Q} = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$H_{\rm SYK}$ is similar, and has identical properties, to the SY model.



A fermion can move only by entangling with another fermion: the Hamiltonian has "nothing but entanglement".

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)
A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

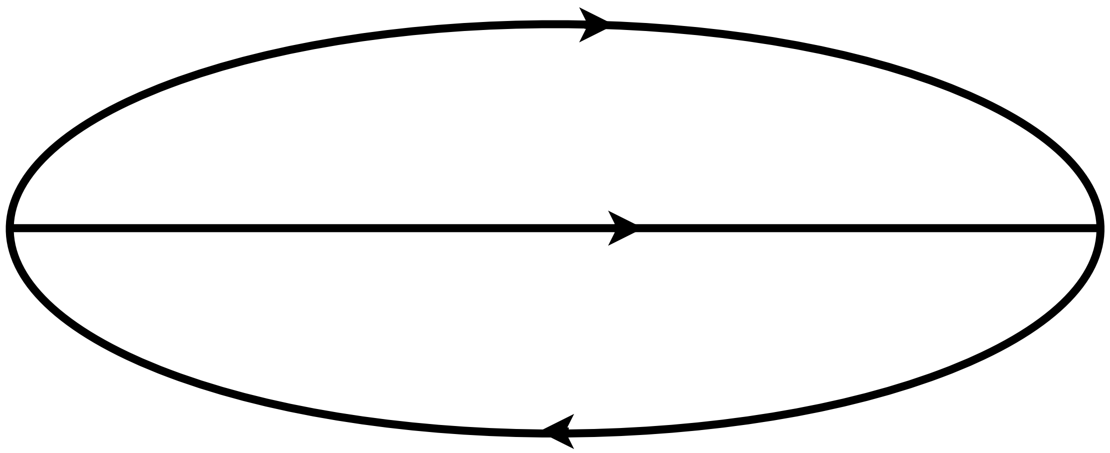
SYK model

Feynman graph expansion in $J_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

$$\Sigma =$$



S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

SYK model

Feynman graph expansion in $J_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$

Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

for some complex A. The ground state is a non-Fermi liquid, with a continuously variable density $\mathcal{Q}$.

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

SYK model

- $T = 0$ Green's function $G \sim 1/\sqrt{\tau}$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

SYK model

- $T = 0$ Green's function $G \sim 1/\sqrt{\tau}$
- $T > 0$ Green's function implies conformal invariance $G \sim 1/(\sin(\pi T \tau))^{1/2}$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)

SYK model

- $T = 0$ Green's function $G \sim 1/\sqrt{\tau}$
- $T > 0$ Green's function implies conformal invariance $G \sim 1/(\sin(\pi T \tau))^{1/2}$
- Non-zero entropy as $T \to 0$, $S(T \to 0) = N S_0 + \ldots$

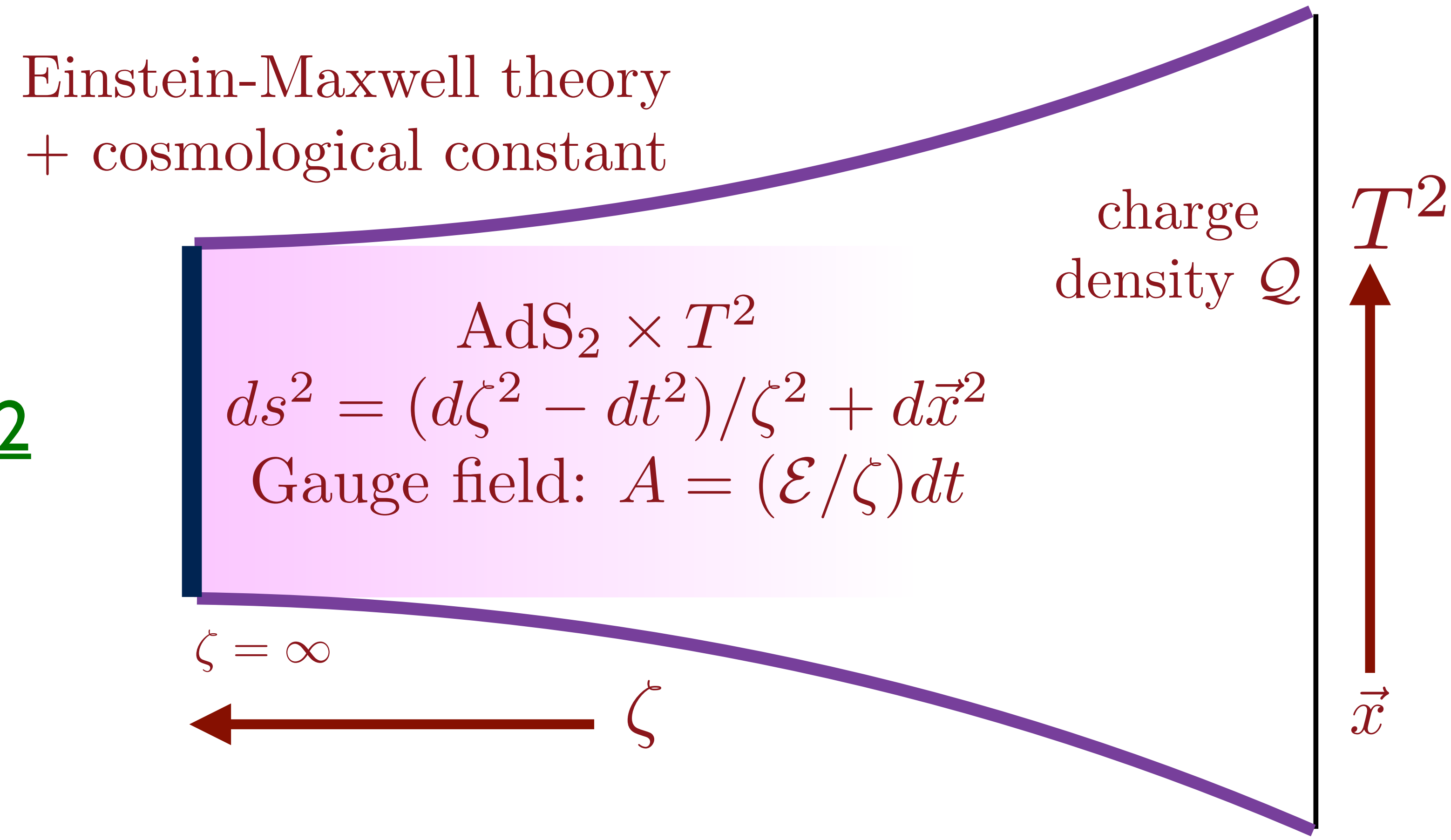
A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

SYK model

- $T = 0$ Green's function $G \sim 1/\sqrt{\tau}$
- $T > 0$ Green's function implies conformal invariance $G \sim 1/(\sin(\pi T \tau))^{1/2}$
- Non-zero entropy as $T \to 0$, $S(T \to 0) = N S_0 + \ldots$
- These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with AdS_2 near-horizon geometry. The Bekenstein-Hawking entropy is $N S_0$. S. Sachdev, PRL **105**, 151602 (2010)
- The dependence of S_0 on the density $\mathcal{Q}$ matches the behavior of the Wald-Bekenstein-Hawking entropy of AdS_2 horizons in a large class of gravity theories.

S. Sachdev, PRX **5**, 041025 (2015)

SYK and AdS_2



PHYSICAL REVIEW LETTERS **105, 151602 (2010)**

Holographic Metals and the Fractionalized Fermi Liquid

Subir Sachdev

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a "small" Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, $AdS_2 \times R^2$ physics of Reissner-Nordström black holes.

SYK model

After integrating the fermions, the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

$$S = \ln \det \left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left[G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

SYK model

After integrating the fermions, the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

$$S = \ln \det \left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left[G(\tau_2, \tau_1) + (J^2/2) G^2(\tau_2, \tau_1) G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll J$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
A. Kitaev, unpublished
S. Sachdev, PRX **5**, 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = \left[f'(\sigma_1) f'(\sigma_2)\right]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = \left[f'(\sigma_1) f'(\sigma_2)\right]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

SYK model

Let us write the large N saddle point solutions of S as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$

These are not invariant under the reparametrization symmetry but are invariant only under a SL(2,R) subgroup under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken.

J. Maldacena and D. Stanford, arXiv:1604.07818
See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768

SYK model

Let us write the large N saddle point solutions of S as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$

Th … in-
var …

So … en.

Connections of SYK to gravity and AdS_2 horizons

- Reparameterization and gauge invariance are the 'symmetries' of the Einstein-Maxwell theory of gravity and electromagnetism
- SL(2,R) is the isometry group of AdS_2.

J. Maldacena and D. Stanford, arXiv:1604.07818
See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768

SYK model

Let us write the large N saddle point solutions of S as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$

These are not invariant under the reparametrization symmetry but are invariant only under a SL(2,R) subgroup under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken.

Reparametrization zero mode

Expand about the saddle point by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1) f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2))$$

(and similarly for Σ) and obtain an effective action for $f(\tau)$. This action does not vanish because of the time derivative in the determinant which is not reparameterization invariant.

J. Maldacena and D. Stanford, arXiv:1604.07818

See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768

SYK model

With $g(\tau) = e^{-i\phi(\tau)}$, the action for $\phi(\tau)$ and $f(\tau) = \dfrac{1}{\pi T}\tan(\pi T(\tau + \epsilon(\tau))$ fluctuations is

$$S_{\phi,f} = \frac{K}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau \epsilon)^2 - \frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau \, \{f, \tau\},$$

where $\{f, \tau\}$ is the Schwarzian:

$$\{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

The couplings are given by thermodynamics (Ω is the grand potential)

$$K = -\left(\frac{\partial^2 \Omega}{\partial \mu^2}\right)_T \quad , \quad \gamma + 4\pi^2 \mathcal{E}^2 K = -\left(\frac{\partial^2 \Omega}{\partial T^2}\right)_\mu$$

$$2\pi \mathcal{E} = \frac{\partial S_0}{\partial \mathcal{Q}}$$

Wenbo Fu, Yingfei Gu, S. Sachdev, unpublished

SYK and AdS_2

With $g(\tau) = e^{-i\phi(\tau)}$, the action for $\phi(\tau)$ and $f(\tau) = \dfrac{1}{\pi T}\tan(\pi T(\tau + \epsilon(\tau))$ fluctuations is

$$S_{\phi,f} = \frac{K}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau \epsilon)^2 - \frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau \, \{f, \tau\},$$

where $\{f, \tau\}$ is the Schwarzian:

$$\{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

- The same effective action is obtained from the Reissner-Nördstrom-AdS black hole of Einstein-Maxwell theory in 4 dimensions, after a dimensional direction to $AdS_2 \times T^2$, valid when the temperature is smaller than a scale set by the size of T^2.

- The Lyapunov time to quantum chaos saturates the lower bound both in the SYK model and in the gravity theory.

$$\tau_L = \frac{1}{2\pi}\frac{\hbar}{k_B T}$$

A. Kitaev, KITP talk, 2015
J. Maldacena and D. Stanford, arXiv:1604.07818
Wenbo Fu, Yingfei Gu, S. Sachdev, unpublished

Entangled quantum matter without quasiparticles

- Is there a connection between strange metals and black holes?
  Yes, *e.g.* the SYK model.

- Why do they have the same equilibration time $\sim \hbar/(k_B T)$?
  Strange metals don't have quasiparticles and thermalize rapidly;
  Black holes are "fast scramblers".

- Theoretical predictions for strange metal transport in graphene agree well with experiments